RADIATIVE TRANSFER EQUATION IN NONINERTIAL COORDINATE SYSTEMS

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Equations are presented describing the interaction of radiation with matter in conditions of arbitrarily large nonequilibrium, both in a local satellite system within the framework of the special theory of relativity, and also in a satellite system based on generalized covariant formalism. The effects of interaction of the radiation generated with the moving material have been correctly accounted for. Calculations have been carried out for the case of a single space coordinate; generalizations to the three-dimensional case is quite easy. In [1, 3] the motion was computed in a class of inertial systems and an analysis determined the important effects associated with acceleration and spatial variation of velocity, very typical for shock waves. In [1,4] it was established how far one may calculate local noninertia in conditions close to equilibrium. An approximate derivation of the analogous equations for the case of local noninertia was given in [5].

1. Local Satellite Systems. Since we intend below to proceed to the case of a satellite system, we choose a metric of two-dimensional space and time in the form very frequently used in the general theory of relativity

$$ds^2 = -(dx^1)^2 + (d\tau)^2. \tag{1.1}$$

Here ds is the interval between events, dx^1 is the element of length in the laboratory system L_0, and $\tau = dx^4 = cdt$ is the product of the time interval in the same system, and the speed of light. Then the contravariant components of the particle 4-velocity in the system L_0 take the form

$$u^1 = \frac{\partial x^1}{\partial s} = \beta\theta, \qquad u^4 = \frac{\partial \tau}{\partial s} = \theta, \qquad \theta = \frac{1}{\sqrt{1-\beta^2}}. \tag{1.2}$$

where β is the ratio of particle speed to the speed of light. It follows from Eq. (1.1) that the metric tensor g_{ik} in system L_0 takes the form

$$g_{ik} = g^{ik} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.3}$$

Thus, the spatial covariant components of all the 4-vectors will differ in sign from the covariant components, while the time components will agree in sign. For example, $u_1 = -u^1$, $u_4 = u^4$. The relation

$$(u^4)^2 - (u^1)^2 = 1. \tag{1.4}$$

also follows from Eqs. (1.1) and (1.2)

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The local stellite system S_0 with spatial coordinate $\xi_0{}^1$ and time coordinate $\tau_0 = \xi_0{}^4 = ct_0$ is related to the laboratory system L_0 by Lorentz transformations for differential coordinates

$$d\xi_0{}^1 = u^4 dx^1 - u^1 d\tau, \qquad d\tau_0 = -u^1 dx^1 + u^4 d\tau. \tag{1.5}$$

It is easy to see that the metric tensor retains the form of Eq. (1.3) in S_0. From Eq. (1.5) the following derivatives can be found

$$\frac{\partial \xi_0{}^1}{\partial x^1} = u^4, \quad \frac{\partial \xi_0{}^1}{\partial \tau} = -u^1, \quad \frac{\partial \tau_0}{\partial x^1} = -u^1, \quad \frac{\partial \tau_0}{\partial \tau} = u^4. \tag{1.6}$$

From the inverse formulas

$$dx^1 = u^4 d\xi_0{}^1 + u^1 d\tau_0, \qquad d\tau = u^1 d\xi_0{}^1 + u^4 d\tau_0, \tag{1.7}$$

we find

$$\frac{\partial x^1}{\partial \xi_0{}^1} = u^4, \quad \frac{\partial x^1}{\partial \tau_0} = u^1, \quad \frac{\partial \tau}{\partial \xi_0{}^1} = u^1, \quad \frac{\partial \tau}{\partial \tau_0} = u^4. \tag{1.8}$$

Formulas (1.6) and (1.8) define elements of the transformation matrix for tensor components. If q^i is a 4-vector, and $q_x{}^i$, q^{ij} are tensors of second rank, the transfer formulas will have the form (the subscript 0 denotes the component of a tensor defined in the system S_0):

$$q_0{}^i = q^\alpha \frac{\partial \xi_0{}^i}{\partial x^\alpha}, \qquad q^i = q_0{}^\alpha \frac{\partial x^i}{\partial \xi_0{}^\alpha}, \qquad q_{0i} = q_\alpha \frac{\partial x^\alpha}{\partial \xi_0{}^i}, \qquad q_i = q_{0\alpha} \frac{\partial \xi_0{}^\alpha}{\partial x^i},$$

$$q_{0\alpha}^{i} = q_s{}^r \frac{\partial \xi_0{}^i}{\partial x^r} \frac{\partial x^s}{\partial \xi_0{}^\alpha}$$

$$q_\alpha{}^i = q_{0s}{}^r \frac{\partial x^i}{\partial \xi_0{}^r} \frac{\partial \xi_0{}^s}{\partial x^\alpha}, \qquad q_0{}^{ij} = q^{rs} \frac{\partial \xi_0{}^i}{\partial x^r} \frac{\partial \xi_0{}^j}{\partial x^s}, \qquad q^{ij} = q_0{}^{rs} \frac{\partial x^i}{\partial \xi_0{}^r} \frac{\partial x^j}{\partial \xi_0{}^s}. \tag{1.9}$$

Using these formulas, it is easy to obtain the relation between the differential operators in the various coordinate systems

$$\frac{\partial}{\partial \tau} = u^4 \frac{\partial}{\partial \tau_0} - u^1 \frac{\partial}{\partial \xi_0{}^1}, \quad \frac{\partial}{\partial x^1} = -u^1 \frac{\partial}{\partial \tau_0} + u^4 \frac{\partial}{\partial \xi_0{}^1}, \quad \frac{\partial}{\partial \tau_0} = u^4 \frac{\partial}{\partial \tau} + u^1 \frac{\partial}{\partial x^1}, \quad \frac{\partial}{\partial \xi_0{}^1} = u^1 \frac{\partial}{\partial \tau} + u^4 \frac{\partial}{\partial x^1}. \tag{1.10}$$

By applying formulas (1.10), it is easy to obtain a relation between the spatial and timewise derivatives of the 4-velocity components in the various systems

$$\frac{\partial u_0{}^1}{\partial \xi_0{}^1} = \frac{\partial u^4}{\partial \tau} + \frac{\partial u^1}{\partial x^1}, \quad \frac{\partial u_0{}^1}{\partial \tau_0} = \frac{\partial u^1}{\partial \tau} + \frac{\partial u^4}{\partial x^1}, \quad \frac{\partial u_0{}^4}{\partial \xi_0{}^1} = 0, \quad \frac{\partial u_0{}^4}{\partial \tau_0} = 0. \tag{1.11}$$

It can be seen from these formulas that a characteristic of a local satellite system S_0, in addition to the fact that $u_0{}^1 = 0$, $u_0{}^4 = 1$, is that the tangent vector to the world line of a particle is constant there.

Similarly,

$$\frac{\partial u^1}{\partial x^1} = (u^4)^2 \frac{\partial u_0{}^1}{\partial \xi_0{}^1} - u^1 u^4 \frac{\partial u_0{}^1}{\partial \tau_0}, \quad \frac{\partial u^1}{\partial \tau} = -u^1 u^4 \frac{\partial u_0{}^1}{\partial \xi_0{}^1} + (u^4)^2 \frac{\partial u_0{}^1}{\partial \tau_0}$$

$$\frac{\partial u^4}{\partial x^1} = u^1 u^4 \frac{\partial u_0{}^1}{\partial \xi_0{}^1} - (u^1)^2 \frac{\partial u_0{}^1}{\partial \tau_0}, \quad \frac{\partial u^4}{\partial \tau} = -(u^1)^2 \frac{\partial u_0{}^1}{\partial \xi_0{}^1} + u^1 u^4 \frac{\partial u_0{}^1}{\partial \tau_0}. \tag{1.12}$$

Now it is easy to obtain the equations of radiative hydrodynamics in system S_0. The tensor for the energy and momentum of an ideal fluid, as is well known, is given by the formula

$$T^{ik} = (p + \varepsilon) u^i u^k - g^{ik} p \tag{1.13}$$

where the internal energy ε and the pressure p are scalars.

The equations of conservation of momentum and energy are

$$\frac{\partial T^{ik}}{\partial x^k} = q^i. \tag{1.14}$$

Here q^i is an 4-vector for the energy and momentum transferred by radiation from the material. The tensor relation between the components has the form

$$\begin{aligned} T^{11} &= T_0^{11}(u^4)^2 + T_0^{44}(u^1)^2, \qquad & T^{14} &= (T_0^{11} + T_0^{44})u^1u^4 \\ T^{44} &= T_0^{11}(u^1)^2 + T_0^{44}(u^4)^2, \qquad & T_0^{11} &= p, \ T_0^{44} = \varepsilon \\ q^4 &= q_0^1u^1 + q_0^4u^4, \qquad & q^1 &= q_0^1u^4 + q_0^4u^1. \end{aligned} \tag{1.15}$$

Therefore the momentum equation in S_0 can be obtained by replacing Eqs. (1.15), (1.10) and (1.12), in Eq. (1.14); the result is that the equation

$$\frac{\partial T^{14}}{\partial \tau} + \frac{\partial T^{11}}{\partial x^1} = q^1$$

transforms to the equation

$$u^4\frac{\partial p}{\partial \xi_0^1} + u^1\frac{\partial \varepsilon}{\partial \tau_0} + (p+\varepsilon)\left[u^4\frac{\partial u_0^1}{\partial \tau_0} + u^1\frac{\partial u_0^1}{\partial \xi_0^1}\right] = u^4q_0^1 + u^1q_0^4. \tag{1.16}$$

Similarly, the energy equation

$$\frac{\partial T^{44}}{\partial \tau} + \frac{\partial T^{14}}{\partial x^1} = q^4$$

transforms to

$$u^1\frac{\partial p}{\partial \xi_0^1} + u^4\frac{\partial \varepsilon}{\partial \tau_0} + (p+\varepsilon)\left[u^1\frac{\partial u_0^1}{\partial \tau_0} + u^4\frac{\partial u_0^1}{\partial \xi_0^1}\right] = u^1q_0^1 + u^4q_0^4. \tag{1.17}$$

Simultaneous solution of Eqs. (1.16) and (1.17) gives

$$\frac{\partial p}{\partial \xi_0^1} + (p+\varepsilon)\frac{\partial u_0^1}{\partial \tau_0} = q_0^1, \qquad \frac{\partial \varepsilon}{\partial \tau_0} + (p+\varepsilon)\frac{\partial u_0^1}{\partial \xi_0^1} = q_0^4. \tag{1.18}$$

These equations differ from those in [6] only in that the right sides are nonzero.

The continuity equation in system S_0 takes the well-known form

$$\frac{\partial \rho}{\partial \tau_0} + \rho\frac{\partial u_0^1}{\partial \xi_0^1} = 0. \tag{1.19}$$

The radiative transfer equations in system L_0 have the form

$$\frac{\partial W^{ik}}{\partial x^k} = -q^i \tag{1.20}$$

where W^{ik} is the radiation energy and momentum tensor, and the vector q^i is defined above. From formula (1.19) we can obtain the relations

$$\begin{aligned} W^{44} &= W_0^{44}(u^4)^2 + 2u^1u^4W_0^{14} + (u^1)^2W_0^{11} \\ W^{14} &= W_0^{44}u^1u^4 + W_0^{14}(u^4u^4 + u^1u^1) + u^1u^4W_0^{11} \\ W^{11} &= W_0^{44}(u^1)^2 + 2u^1u^4W_0^{14} + (u^4)^2W_0^{11}\,. \end{aligned} \tag{1.21}$$

Using Eq. (1.21) and the foregoing procedure to obtain formulas in system S_0, we find that the equations of radiative energy and momentum take the form

$$\begin{aligned} \frac{\partial W_0^{44}}{\partial \tau_0} + \frac{\partial W_0^{14}}{\partial \xi_0^1} + (W_0^{44} + W_0^{11})\frac{\partial u_0^1}{\partial \xi_0^1} + 2W_0^{14}\frac{\partial u_0^1}{\partial \tau_0} &= -q_0^4 \\ \frac{\partial W_0^{14}}{\partial \tau_0} + \frac{\partial W_0^{11}}{\partial \xi_0^1} + (W_0^{44} + W_0^{11})\frac{\partial u_0^1}{\partial \tau_0} + 2W_0^{14}\frac{\partial u_0^1}{\partial \xi_0^1} &= -q_0^1\,. \end{aligned} \tag{1.22}$$

These equations can be reduced to tensor form, which permits generalization to the multi-dimensional case:

$$\frac{\partial W_0^{ik}}{\partial \xi_0^k} + \left(W_0^{kj}u_{0k}\frac{\partial u_0^i}{\partial \xi_0^j} - W_0^{kj}u_0^i\frac{\partial u_{0k}}{\partial \xi_0^j}\right) + \left(W_0^{ik}u_{0k}\frac{\partial u_0^j}{\partial \xi_0^j} - W_0^{ik}u_0^j\frac{\partial u_{0k}}{\partial \xi_0^j}\right) = -q_0^i\,. \tag{1.23}$$

The 4-vector for transfer of material energy and momentum into radiation, allowing for scattering effects, was obtained earlier in [7]

$$-q_0^i = \alpha_0 u_0^i\,\frac{4\pi}{c}\,B_0 + \sigma_0 u_0^i u_{0k}u_{0j}W_0^{kj} - (\alpha_0 + \sigma_0)\,u_{0k}W_0^{ik}\,. \tag{1.24}$$

Here B_0 is the Planck function integrated over frequency; α_0, σ_0 are linear coefficients for radiation absorption and scattering by the undisturbed substance respectively. Using Eqs. (1.18) and (1.22), we can write laws for conservation of the total tensor of energy and momentum of matter and radiation

$$\frac{\partial}{\partial x^k}\,(T^{ik} + W^{ik}) = 0$$

in the system S_0, in the form

$$\begin{aligned} \frac{\partial}{\partial \tau_0}(\varepsilon + W_0^{44}) + \frac{\partial W_0^{14}}{\partial \xi_0^1} + (p + \varepsilon + W_0^{11} + W_0^{44})\frac{\partial u_0^1}{\partial \xi_0^1} + 2W_0^{14}\frac{\partial u_0^1}{\partial \tau_0} &= 0 \\ \frac{\partial}{\partial \xi_0^1}(p + W_0^{11}) + \frac{\partial W_0^{14}}{\partial \tau_0} + (p + \varepsilon + W_0^{11} + W_0^{44})\frac{\partial u_0^1}{\partial \tau_0} + 2W_0^{14}\frac{\partial u_0^1}{\partial \xi_0^1} &= 0\,. \end{aligned} \tag{1.25}$$

These equations, along with the transfer equation (1.23) and the continuity equation (1.19), constitute a complete system describing the interaction of radiation with matter in a local satellite coordinate system. It is not a closed system. To close it we must introduce an additional relation between the radiation energy and momentum tensor components. It is very convenient to obtain Eqs. (1.22) by another method, directly from the transfer equation. Besides being a proof of the validity of the formula, this method enables us to follow in detail the physical nature of the additional terms arising, and also to obtain formulas for radiative viscosity in the case of weak nonequilibrium.

As was shown in [7,8], the transfer equation for the integrated intensity of radiation I in the laboratory system L_0, allowing for scattering processes, has the form

$$\left(\frac{\partial}{\partial \tau} + \mu\,\frac{\partial}{\partial x^1}\right)I = -(\alpha_0 + \sigma_0)IL + \alpha_0\frac{B_0}{L^3} + \frac{\sigma_0}{AL^3}\,[(u^4)^2W^{44} - 2u^1u^4W^{14} + (u^1)^2W^{11}] \tag{1.26}$$

$$L = u^4 - \mu u^1, \qquad A = \frac{8\pi}{c\rho_0 v_0^2}$$

$$W^{11} = \frac{A}{2}\int_{-1}^{1} I\mu^2 d\mu, \qquad W^{14} = \frac{A}{2}\int_{-1}^{1} I\mu d\mu, \qquad W^{44} = \frac{A}{2}\int_{-1}^{1} Id\mu\,. \tag{1.27}$$

Here μ is the cosine of the angle between the direction of motion of the material and the direction of the incident radiative quantum, and ρ_0 and ν_0 are some characteristic values of material density and velocity. In conversion to a local satellite system S_0 in which the material element is at rest, the components of the radiative energy and momentum tensor W^{ik} are transformed according to formulas analogous to (1.21), while the remaining quantities are transformed according to the following formulas [2]:

$$I = \frac{I_0}{L^4}, \quad \mu = \frac{\mu_0 u^4 + u^1}{u^4 + \mu_0 u^1}, \quad \mu_0 = \frac{\mu u^4 - u^1}{u^4 - \mu u^1}, \quad d\mu = L^2 d\mu_0$$

$$L = u^4 - \mu u^1 = \frac{1}{u^4 + \mu u^1}, \quad \frac{\partial}{\partial \tau} + \mu \frac{\partial}{\partial x^1} = L\left(\frac{\partial}{\partial \tau_0} + \mu_0 \frac{\partial}{\partial \xi_0^1}\right). \tag{1.28}$$

Using these formulas, we can find that in system S_0, the transfer equation takes the form

$$L^4 \left(\frac{\partial}{\partial \tau_0} + \mu_0 \frac{\partial}{\partial \xi_0^1}\right) \frac{I_0}{L^4} = -(\alpha_0 + \sigma_0) I_0 + \alpha_0 B_0 + \frac{\sigma_0}{A} W_0^{44}. \tag{1.29}$$

Using Eqs. (1.12), it is easy to obtain the derivative with respect to L

$$\left(\frac{\partial}{\partial \tau_0} + \mu_0 \frac{\partial}{\partial \xi_0^1}\right) L = \frac{1}{L}\left(\frac{\partial}{\partial \tau} + \mu \frac{\partial}{\partial x^1}\right) L = \frac{1}{L}\left[\left(\frac{\partial}{\partial \tau} + \mu \frac{\partial}{\partial x^1}\right) u^4 - \mu \left(\frac{\partial}{\partial \tau} + \mu \frac{\partial}{\partial x^1}\right) u^1\right] = -L\mu_0 \left(\frac{\partial u_0^1}{\partial \tau_0} + \mu_0 \frac{\partial u_0^1}{\partial \xi_0^1}\right). \tag{1.30}$$

Therefore, Eq. (1.29) can be rewritten in a more detailed form

$$\frac{\partial I_0}{\partial \tau_0} + \mu_0 \frac{\partial I_0}{\partial \xi_0^1} + 4 I_0 \mu_0 \left(\frac{\partial u_0^1}{\partial \tau_0} + \mu_0 \frac{\partial u_0^1}{\partial \xi_0^1}\right) = -(\alpha_0 + \sigma_0) I_0 + \alpha_0 B_0 + \frac{\sigma_0}{A} W_0^{44}. \tag{1.31}$$

Since the quantities W_0^{ik} appear in the equations of radiative hydrodynamics, we must integrate Eq. (1.31) with respect to $d\mu_0$. However, we must take into account that the differential $d\mu_0$ cannot pass under the differentiation sign, since the differential $d\mu$ is a fixed quantity. Clearly we must use a more complex procedure, this being

$$d\mu_0 (\nabla_0 \Phi_0) = \nabla_0 (\Phi_0 d\mu_0) - \Phi_0 \nabla_0 (d\mu_0) = \nabla_0 (\Phi_0 d\mu_0) - \Phi_0 d\mu \nabla_0 (1/L^2) = \nabla_0 (\Phi_0 d\mu_0) - 2\Phi_0 \mu_0 d\mu_0 \nabla_0 u_0^1. \tag{1.32}$$

Here Φ_0 is an arbitrary function of μ_0, ξ_0^1 and τ_0, and ∇_0 is a spatial or timewise linear differential operator. In a similar way, using the invariance of μ and $d\mu$, we obtain

$$\mu_0 \nabla_0 \Phi_0 = \nabla_0 (\mu_0 \Phi_0) + \Phi_0 (1 - \mu_0^2) \nabla_0 u_0^1. \tag{1.33}$$

Using these rules, we can multiply Eq. (1.31) by $d\mu_0$ and $\mu_0 d\mu_0$, and reduce it to a form convenient for conversion to integral quantities. The result is the two equations

$$\frac{\partial}{\partial \tau_0}(I_0 d\mu_0) + \frac{\partial}{\partial \xi_0^1}(I_0 \mu_0 d\mu_0) + 2 I_0 \mu_0 d\mu_0 \frac{\partial u_0^1}{\partial \tau_0} + I_0 d\mu_0 \frac{\partial u_0^1}{\partial \xi_0^1} + I_0 \mu_0^2 d\mu_0 \frac{\partial u_0^1}{\partial \xi_0^1} = -(\alpha_0 + \sigma_0) I_0 d\mu_0 + \alpha_0 B_0 d\mu_0 + \frac{\sigma_0}{A} W_0^{44} d\mu_0$$

$$\frac{\partial}{\partial \tau_0}(I_0 \mu_0 d\mu_0) + \frac{\partial}{\partial \xi_0^1}(I_0 \mu_0^2 d\mu_0) + I_0 (1 + \mu_0^2) d\mu_0 \frac{\partial u_0^1}{\partial \tau_0} + 2 I_0 \mu_0 d\mu_0 \frac{\partial u_0^1}{\partial \xi_0^1} = -(\alpha_0 + \sigma_0) I_0 \mu_0 d\mu_0 + \alpha_0 B_0 \mu_0 d\mu_0 + \frac{\sigma_0}{A} W_0^{44} \mu_0 d\mu_0. \tag{1.34}$$

Converting to components W_0^{ik} using formulas analogous to (1.27), we obtain Eq. (1.22)

$$\frac{\partial W_0^{44}}{\partial \tau_0} + \frac{\partial W_0^{14}}{\partial \xi_0^1} + 2 W_0^{14} \frac{\partial u_0^1}{\partial \tau_0} + (W_0^{44} + W_0^{11}) \frac{\partial u_0^1}{\partial \xi_0^1} = \alpha_0 (A B_0 - W_0^{44}) = -q_0^4$$

$$\frac{\partial W_0^{14}}{\partial \tau_0} + \frac{\partial W_0^{11}}{\partial \xi_0^1} + (W_0^{44} + W_0^{11}) \frac{\partial u_0^1}{\partial \tau_0} + 2 W_0^{14} \frac{\partial u_0^1}{\partial \xi_0^1} = -(\alpha_0 + \sigma_0) W_0^{14} = -q_0^1. \tag{1.35}$$

By the above method we can obtain equations also for the higher moments. For example, introducing the moments of third and fourth order

$$M_0 = \frac{A}{2}\int_{-1}^{1} I_0\mu_0^3 d\mu_0, \qquad N_0 = \frac{A}{2}\int_{-1}^{1} I_0\mu_0^4 d\mu_0, \tag{1.36}$$

and multiplying Eq. (1.31) by $\mu_0^2 d\mu_0$, we can similarly obtain the equation

$$\frac{\partial W_0^{11}}{\partial \tau_0} + \frac{\partial M_0}{\partial \xi_0^1} + 2W_0^{14}\frac{\partial u_0^1}{\partial \tau_0} + 3W_0^{11}\frac{\partial u_0^1}{\partial \xi_0^1} - N_0\frac{\partial u_0^1}{\partial \xi_0^1} = -(\alpha_0+\sigma_0)W_0^{11} + \alpha_0\frac{AB_0}{3} + \frac{\sigma_0 W_0^{44}}{3}. \tag{1.37}$$

Equations (1.35) and (1.37) for static conditions ($\nabla_0 = 0$), give the following values of the moments:

$$W_0^{44} = AB_0, \quad W_0^{14} = 0, \quad W_0^{11} = {}^1/_3 AB_0.$$

These values can be obtained for M_0 and N_0 by integrating the right sides of Eqs. (1.34) with appropriate multipliers

$$M_0 = 0, \qquad N_0 = {}^1/_5 AB_0.$$

Taking these values of the moments as a zeroth approximation, from Eqs. (1.35) and (1.37) we obtain expressions coinciding with those of [3] with $\sigma_0 = 0$ for the first approximation to the three main moments

$$\begin{aligned}
\frac{1}{A}\overline{W}_0^{44} &= B_0 - \frac{1}{\alpha_0}\frac{\partial B_0}{\partial \tau_0} - \frac{4B_0}{3\alpha_0}\frac{\partial u_0^1}{\partial \xi_0^1}\\
\frac{1}{A}\overline{W}_0^{14} &= -\frac{1}{3(\alpha_0+\sigma_0)}\frac{\partial B_0}{\partial \xi_0^1} - \frac{4B_0}{3(\alpha_0+\sigma_0)}\frac{\partial u_0^1}{\partial \tau_0}\\
\frac{1}{A}\overline{W}_0^{11} &= \frac{1}{3}B_0 - \frac{1}{3(\alpha_0+\sigma_0)}\frac{\partial B_0}{\partial \tau_0} - \frac{4B_0}{5(\alpha_0+\sigma_0)}\frac{\partial u_0^1}{\partial \xi_0^1}\left(\frac{5\sigma_0}{9\alpha_0}+1\right).
\end{aligned} \tag{1.38}$$

The last term in the expression for $\overline{W}_0^{11}$ is interpreted as the radiative viscosity. All the above equations retain their form if their derivation begins, not from Eq. (1.26), but from the equation for the spectral radiative intensity I_ν.

2. Satellite System. To define a satellite we use the formalism developed in the work of L. I. Sedov [9]. We introduce two coordinate systems, an inertial system $K(x^i, x^4 = \tau)$ with metric (1.1), and a moving satellite system $L(\xi^{[i]}, \xi^{[4]})$ with metric

$$ds^2 = g_{[ik]} d\xi^{[i]} d\xi^{[k]} \tag{2.1}$$

Coordinates x^i, $\xi^{[i]}$ are chosen in a single pseudo-Euclidean space, i.e., functional relations $x^i = x^i(\xi^{[i]}, \xi^{[4]})$ exist which define the law of motion of the continuum under consideration. This law can be defined in explicit form if we take into account that the spatial coordinate $\xi^{[i]}$ of the system L coincides with the Cartesian coordinate x^i of system K at some time instant $\xi^{[4]} = \xi^*$ in the entire three-dimensional space, while the differentials of the self-times are connected by the Lorentz relations for coincident points, i.e., for the relation between the coordinates of these systems we obtain, with $\xi^{[4]} = \xi^*$,

$$dx^1 = d\xi^{[1]} + u^1 d\xi^{[4]}, \qquad d\tau = u^4 d\xi^{[4]} \tag{2.2}$$

where u^i is the 4-velocity of the points of system L relative to K. Thus, system K is some fixed position of the deformed continuum relative to which the system L moves, described by the fact that the 4-velocity of points determined relative to it is given by the expressions

$$u^{[1]} = \frac{d\xi^{[1]}}{ds} = 0, \qquad u^{[4]} = \frac{d\xi^{[4]}}{ds} = 1 \tag{2.3}$$

Substitution of Eqs. (2.2) into the definition (2.1) enables us to obtain the metric tensor $g_{[ik]}$

$$g_{[ik]} = \begin{pmatrix} -1 & -u^1 \\ -u^1 & 1 \end{pmatrix}, \quad g^{[ik]} = \begin{pmatrix} -1/u_4^2 & -u^1/u_4^2 \\ -u^1/u_4^2 & 1/u_4^2 \end{pmatrix} \tag{2.4}$$

In the system L all the differential operations must be determined in a covariant manner, for which we must calculate the Christoffel symbols

$$\Gamma^{\nu}_{\alpha\beta} = \frac{\partial^2 x^i}{\partial \xi^{[\alpha]}\partial \xi^{[\beta]}}\frac{\partial \xi^{[\nu]}}{\partial x^i} \tag{2.5}$$

The first derivatives are determined from Eqs. (2.2) and the inverse relations

$$\frac{\partial x^1}{\partial \xi^{[1]}} = 1, \quad \frac{\partial x^1}{\partial \xi^{[4]}} = u^1, \quad \frac{\partial \tau}{\partial \xi^{[4]}} = u^4, \quad \frac{\partial \tau}{\partial \xi^{[1]}} = 0$$
$$\frac{\partial \xi^{[1]}}{\partial x^1} = 1, \quad \frac{\partial \xi^{[1]}}{\partial \tau} = -\frac{u^1}{u^4}, \quad \frac{\partial \xi^{[4]}}{\partial x^1} = 0, \quad \frac{\partial \xi^{[4]}}{\partial \tau} = \frac{1}{u^4} \tag{2.6}$$

The second derivatives appearing in Eq. (2.5) are calculated in a somewhat more complex way, especially the mixed derivatives, for example,

$$\left.\frac{\partial^2 x^1}{\partial \xi^{[4]}\partial \xi^{[1]}}\right|_{\xi^*} = \frac{1}{d\xi^*}\left[\left.\frac{\partial x^1}{\partial \xi^{[1]}}\right|_{\xi^*+d\xi^*} - \left.\frac{\partial x^1}{\partial \xi^{[1]}}\right|_{\xi^*}\right] =$$
$$= \frac{1}{d\xi^{[*]}}\left[\frac{\partial (x^1 + u^1 d\xi^*)}{\partial \xi^{[1]}} - \frac{\partial x^1}{\partial \xi^{[1]}}\right] = \frac{\partial u^1}{\partial \xi^{[1]}}$$
$$\left.\frac{\partial^2 \tau}{\partial \xi^{[4]}\partial \xi^{[1]}}\right|_{\xi^*} = \frac{1}{d\xi^*}\left[\left.\frac{\partial \tau}{\partial \xi^{[1]}}\right|_{\xi^*+d\xi^*} - \left.\frac{\partial \tau}{\partial \xi^{[1]}}\right|_{\xi^*}\right] = \frac{1}{d\xi^*}\left[\frac{\partial (u^4 d\xi^*)}{\partial \xi^{[1]}}\right] = \frac{\partial u^4}{\partial \xi^{[1]}}$$

The result is

$$\frac{\partial^2 x^1}{\partial \xi^{[4]2}} = \frac{\partial u^1}{\partial \xi^{[4]}}, \quad \frac{\partial^2 x^1}{\partial \xi^{[4]}\partial \xi^{[1]}} = \frac{\partial u^1}{\partial \xi^{[1]}}, \quad \frac{\partial^2 x^1}{\partial \xi^{[1]2}} = 0,$$
$$\frac{\partial^2 \tau}{\partial \xi^{[4]2}} = \frac{\partial u^4}{\partial \xi^{[4]}}, \quad \frac{\partial^2 \tau}{\partial \xi^{[1]}\partial \xi^{[4]}} = \frac{\partial u^4}{\partial \xi^{[1]}}, \quad \frac{\partial^2 \tau}{\partial \xi^{[1]2}} = 0 \tag{2.7}$$

In Eq. (2.6), as in Eq. (2.7) all the derivatives are taken for $\xi^{[4]} = \xi^*$. Using these formulas, it is easy to obtain

$$\Gamma^1_{11} = 0, \quad \Gamma^1_{14} = \frac{1}{u_4^2}\frac{\partial u^1}{\partial \xi^{[1]}}, \quad \Gamma^1_{44} = \frac{1}{u_4^2}\frac{\partial u^1}{\partial \xi^{[4]}}$$
$$\Gamma^4_{11} = 0, \quad \Gamma^4_{14} = \frac{1}{u_4}\frac{\partial u^4}{\partial \xi^{[1]}}, \quad \Gamma^4_{44} = \frac{1}{u_4}\frac{\partial u^4}{\partial \xi^{[4]}}. \tag{2.8}$$

To obtain relations analogous to (1.12), we can use the relations $u^1 \nabla u^1 = u^4 \nabla u^4$, following from Eq. (1.4). Here ∇ is an arbitrary differential operator. Then, introducing symbols for the acceleration and deformation factors, we obtain

$$\Gamma^1_{14} = D, \quad \Gamma^1_{14} = u^1 D, \quad \Gamma^1_{14} = F, \; \Gamma^4_{44} = u^1 F$$
$$D = \frac{1}{u_4^2}\frac{\partial u^1}{\partial \xi^{[1]}}, \quad F = \frac{1}{u_4^2}\frac{\partial u^1}{\partial \xi^{[4]}}. \tag{2.9}$$

Since Eq. (2.3), $u^{[4]}\nabla u^{[4]} = u^{[1]}\nabla u^{[1]} = 0$, from the definition of a covariant derivative,

$$u^i_{;l} = \frac{\partial u^{[i]}}{\partial \xi^{[l]}} + \Gamma^i_{kl} u^{[k]} \frac{\partial u^{[l]}}{\partial \xi^{[l]}} + \Gamma^i_{4l}$$

we find the relations

$$\frac{\partial u^1}{\partial x^1} = \frac{\partial u^{[1]}}{\partial \xi^{[1]}} + D, \quad \frac{\partial u^1}{\partial \tau} = \frac{\partial u^{[1]}}{\partial \xi^{[4]}} + F, \quad \frac{\partial u^4}{\partial x^1} = u^1 D, \quad \frac{\partial u^4}{\partial \tau} = u^1 F. \tag{2.10}$$

Using the relations $u^1 \nabla u^1 = u^4 \nabla u^4$, we have

$$\frac{\partial u^{[1]}}{\partial \xi^{[1]}} = D(u^4 - 1), \quad \frac{\partial u^{[1]}}{\partial \xi^{[4]}} = F(u^4 - 1) \tag{2.11}$$

and therefore, it follows from Eq. (2.10) that

$$\frac{\partial u^1}{\partial x^1} = u^4 \frac{\partial u^{[1]}}{\partial \xi^{[1]}}, \quad \frac{\partial u^1}{\partial \tau} = u^4 \frac{\partial u^{[1]}}{\partial \xi^{[4]}}, \quad \frac{\partial u^4}{\partial x^1} = \frac{u^1}{u^4 - 1} \times \frac{\partial u^{[1]}}{\partial \xi^{[1]}}, \quad \frac{\partial u^4}{\partial \tau} = \frac{u^1}{u^4 - 1}\frac{\partial u^{[1]}}{\partial \xi^{[4]}}. \tag{2.12}$$

These relations enable us to represent the coefficients D and F in any desired form

$$D = \frac{1}{u_4^2}\frac{\partial u^1}{\partial \xi^{[1]}} = \frac{1}{u^4-1}\frac{\partial u^{[1]}}{\partial \xi^{[1]}} = \frac{1}{u^4(u^4-1)}\frac{\partial u^1}{\partial x^1},$$
$$F = \frac{1}{u_4^2}\frac{\partial u^1}{\partial \xi^{[4]}} = \frac{1}{u^4-1}\frac{\partial u^{[1]}}{\partial \xi^{[4]}} = \frac{1}{u^4(u^4-1)}\frac{\partial u^1}{\partial \tau}. \tag{2.13}$$

In converting to system L all the tensor and vector quantities must be transformed in accordance with Eq. (1.9), using Eq. (2.6).

The result is easily obtained as

$$W^{11} = W^{[11]} + 2u^1W^{[14]} + (u^1)^2W^{[44]},\quad W^{14} = u^4W^{[14]} + u^1u^4W^{[44]},\quad W^{44} = (u^4)^2W^{[44]}, q^1 = q^{[1]} + u^1q^{[4]},\ q^4 = u^4q^{[4]}$$

$$W^{[11]} = W^{11} - \frac{2u^1}{u^4}W^{14} + \left(\frac{u^1}{u^4}\right)^2 W^{44},\quad W^{[14]} = \frac{1}{u^4}W^{14} - \frac{u^1}{u^4}W^{44}$$
$$W^{[44]} = \frac{1}{u_4^2}W^{44},\quad q^{[1]} = q^1 - \frac{u^1}{u^4}q^4,\quad q^{[4]} = \frac{1}{u^4}q^4. \tag{2.14}$$

The moment equations in system L have the form

$$\frac{\partial W^{[\alpha k]}}{\partial \xi^{[k]}} + \Gamma^{\alpha}_{mk}W^{[mk]} + \Gamma^{m}_{km}W^{[\alpha k]} = -q^{[\alpha]} \tag{2.15}$$

In expanded form these equations take the following form

$$\frac{\partial W^{[44]}}{\partial \xi^{[4]}} + \frac{\partial W^{[14]}}{\partial \xi^{[1]}} + 3u^1DW^{[14]} + (D + 2u^1F)W^{[44]} = -q^{[4]}$$
$$\frac{\partial W^{[14]}}{\partial \xi^{[4]}} + \frac{\partial W^{[11]}}{\partial \xi^{[1]}} + u^1DW^{[11]} + (3D + u^1F)W^{[14]} + FW^{[44]} = -q^{[1]} \tag{2.16}$$

The hydrodynamic equations will be analogous to Eqs. (2.15) and (2.16), if $W^{[ik]}$ is replaced by $T^{[ik]}$, and $q^{[i]}$ by $(-q^{[i]})$. If we use equations (2.14) or the definition $T^{[ik]} = (p + \varepsilon)u^{[i]}u^{[k]} - g^{[ik]}p$, the tensor components T^{ik} become

$$T^{[11]} = \frac{1}{u_4^2}p,\quad T^{[14]} = \frac{u^1}{u_4^2}p,\quad T^{[44]} = \varepsilon + p\frac{(u^1)^2}{u_4^2} \tag{2.17}$$

By substituting these equations into Eq. (2.16) for $T^{[ik]}$, we obtain the hydrodynamic equations in the satellite system S in the form

$$\frac{\partial \varepsilon}{\partial \xi^{[4]}} + \frac{u^1}{u_4^2}\frac{\partial p}{\partial \xi^{[1]}} + \left(\frac{u^1}{u^4}\right)^2\frac{\partial p}{\partial \xi^{[4]}} + \varepsilon(D + 2u^1F) + p\left[D + 2u^1F + 2D\left(\frac{u^1}{u^4}\right)^2\right] = q^{[4]}$$
$$\frac{u^1}{u_4^2}\frac{\partial p}{\partial \xi^{[4]}} + \frac{1}{u_4^2}\frac{\partial p}{\partial \xi^{[1]}} + \varepsilon F + p\left[F + D\left(\frac{u^1}{u^4}\right)^2\right] = q^{[1]} \tag{2.18}$$

It is not difficult to see that in the special case of a motionless point ($u^1 \equiv 0$, $u^4 \equiv 1$) which corresponds to a local satellite system, we obtain Eq. (1.18). Equations (2.16) and (2.18), together with equations (2.14) and (2.17), describe the interaction of radiation with a nonuniformly moving deformed continuum. To close the system of equations we need to postulate a relation between the radiative energy and momentum tensor components, and the Eddington hypothesis $W^{[11]} = {}^1/_3 W^{[44]}$ is the best-founded hypothesis in system S.

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